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CONSTRUCTION OF SOLUTIONS FOR TWO-DIMENSIONAL RIEMANN PROBLEMS

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Abstract—Solutions to the scalar quasilinear equation

$$\partial u(t, x)/\partial t + \Sigma_{i=1}^{2} \partial f_i(u(t, x))/\partial x_i = 0,$$

for $f_i \in C^2 : R \rightarrow R$ with initial data given by a two-dimensional Riemann problem, are piecewise smooth if $f_1 \equiv f_2 \equiv f$, and f has at most one inflection point. We show that the "pieces" of this solution can be classified and are expressible in terms of two-dimensional nonlinear waves in analogy with the nonlinear rarefaction and shock waves of the Riemann problem in one spatial dimension. The two-dimensional waves can be expressed in almost-closed form. Explicit solutions are constructable from these waves. An application is illustrated by calculation of the interaction of water/oil banks in two-phase incompressible flow in reservoirs.

1. INTRODUCTION

The Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} + \sum_{i=1}^{2} \frac{\partial f_i(u(t, x))}{\partial x_i} = 0, \tag{1.1}$$

with initial data piecewise constant on a finite number of wedges focused on a single point in the x, y plane, is defined as a *two-dimensional Riemann problem*. Without loss of generality, this point can be taken to be the point $x = 0$, $y = 0$. In a previous paper[1] we have shown that, for $f_i : R \rightarrow R$ in C^2, if $f_1 \equiv f_2 \equiv f$, with f having at most one inflection point, the solution to the two-dimensional Riemann problem is piecewise smooth. The proof consists of two parts. First, it is shown that the two-dimensional Riemann problem for $f_1 \equiv f_2 \equiv f$ is equivalent to a generalization of the Riemann problem in one dimension. It is then proven that the generalized one-dimensional Riemann problem is piecewise smooth if f has at most one inflection point. The proof of the smoothness of the solutions to the two-dimensional Riemann problem does not provide a convenient method of constructing its solutions. In this paper we formulate a construction that is purely two dimensional and has the advantage (at least in specific cases) of generalizing to the problem $f_1 \neq f_2$. This construction consists of identifying the general two-dimensional nonlinear waves, analogous to the rarefaction and shock waves of the one-dimensional Riemann problem, and then using the method initiated by Guckenheimer[2] and Wagner[3] to piece these waves together into an entropy-obeying solution.

Existence and uniqueness of solutions to the two-dimensional Riemann problem within the class of bounded, measurable functions is due to Kružkov[4]. For piecewise-smooth solutions, Kružkov's uniqueness condition reduces to two requirements on the jump discontinuities in the solution:

$$n \cdot [u^+ - u^-, f(u^+) - f(u^-)] = 0, \tag{1.2}$$

and

$$n \cdot [k - u^-, f(k - f(u^-)] \geq 0, \tag{1.3}$$

where f stands for the vector $[f_1, f_2]$. In (1.2), the normal n to the discontinuity is oriented such that $u^- \leq u^+$ and k is any constant such that $u^- \leq k \leq u^+$. We refer to [1] for definition of the terms used here and throughout this article.

Equation (1.2) is denoted the jump condition for the discontinuity and (1.3) the entropy condition; in $R^1 \times R^+$ (1.2) is the familiar Rankine–Hugoniot condition and (1.3) is equivalent to the entropy condition of Oleĭnik[5]. The local nature of conditions (1.2) and (1.3) allow construction of piecewise-smooth global solutions by piecing together local solutions which individually obey (1.2) and (1.3).

In Sec. 2, the two-dimensional nonlinear waves are described for the class of problems $f_1 \equiv f_2$. In Sec. 3 a method for explicitly constructing the two-dimensional solutions in this class from the nonlinear waves is described. Complete solutions for two-dimensional Riemann problems for representative forms of the function f are given. Solutions to the problem of water/oil bank interactions in two-phase, incompressible, gravity-free flow in reservoirs can be obtained from the study of these two-dimensional Riemann problems. The results for this physical problem are presented in Sec. 4. Conclusions and conjectures for the solutions to two-dimensional Riemann problems for the case f having more than one inflection points or $f_1 \neq f_2$ are presented in Sec. 5.

2. THE TWO-DIMENSIONAL RIEMANN PROBLEM

For convenience we shall use the notation $f_1 \equiv f$, $f_2 \equiv g$, $x_1 \equiv x$, $x_2 \equiv y$.

A sufficient condition for a two-dimensional Riemann problem to have a piecewise-smooth solution is given by [1].

THEOREM 2.1

The unique (in the sense of Kružkov) solution in the plane $y > x$, to (1.1) with initial data that is piecewise constant on a finite number of wedges focused on a single point in the plane, with $f_1 \equiv f_2 \equiv f$, $f \in C^2 : R \to R$, f having at most one inflection point, is

(a) piecewise smooth;
(b) composed of nonlinear waves and constant states which are the images under a continuous map M of the rarefaction and shock waves and constant states of a generalized Riemann problem in one dimension;
(c) composed of curves of irregular points corresponding to images under the map M of the irregular points in the one-dimensional Riemann problem.

For the case $g \equiv f$, under the 45° rotation $2\xi = x + y$, $2\eta = y - x$, (1.1) becomes

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial \xi} = 0. \tag{2.1}$$

From (2.1) we see that the solution can be obtained along each η = const plane independent of other η. In particular, this implies the solutions obtained in the $\eta < 0$ half-space can be obtained independently of the solutions in the $\eta > 0$ half; by the symmetry of the problem, no new solutions will be found in the $\eta < 0$ halfspace that are not found in the upper. We therefore restrict our discussion to the half-space $\eta > 0$ $(y > x)$. The solution in the plane $\eta = 0$ is given in Lemma 2.2 below (see [1]).

LEMMA 2.2

(a) If the half-line $\eta = 0$, $\xi < 0$ is not a line of discontinuity of the initial data (i.e. is not a wedge line) the waves incident upon the corresponding half-plane $t > 0$, $\xi < 0$, $\eta = 0$ are continuous across the half-plane.
(b) If the half-line $\eta = 0$, $\xi < 0$ is a line of discontinuity (constant jump) of the initial data, it remains a half-plane $t > 0$, $\xi < 0$, $\eta = 0$ of (in general variable) jump discontinuity in the solution.

Similar statements hold for the half-line $\eta = 0$, $\xi > 0$.

Although the map M mentioned in Theorem 2.1 provides a means of constructing the solution to the two-dimensional Riemann problem, it is not a convenient method of doing so. In particular it is a method that will not generalize to cases $f \neq g$, where no appeal to a one-dimensional analysis can be made. We therefore proceed by formulating a construction that is purely two-dimensional and has the advantage of generalizing (at least in specific cases) to the problem $f \neq g$. We propose that the correct method of dealing with the general solution to the two-dimensional Riemann problem is to identify the general two-dimensional nonlinear waves. This construction method (initiated by Guckenheimer[2] and Wagner[3]) can then be used to piece these waves together into an entropy-obeying solution in a manner that places no reliance on one-dimensional analyses.

2.3. *Two-dimensional nonlinear waves for piecewise-smooth solutions*

We proceed with the analysis and definition of rarefaction and shock waves valid for piecewise-smooth, entropy-obeying solutions to the two-dimensional Riemann problem for the case $f \equiv g$. The explicit forms of these waves and the complete characterization of the irregular points are given.

The two-dimensional Riemann problem is invariant under the similarity transformation $(t, x, y) \rightarrow (ct, cx, cy)$. The solution is therefore constant along rays having the space–time origin as the vertex. Consequently the solution can be determined by its restriction to any plane $t = \text{const}\ (>0)$. The plane $t = 1$ is most convenient for this purpose.

In addition to this self-similarity property, at points at which u is regular, the solution is constant on characteristics $(t, x(t), y(t))$ which are the straight lines

$$\frac{x - x_0}{t - t_0} = f'(u), \qquad \frac{y - y_0}{t - t_0} = g'(u). \qquad t_0 = 0. \tag{2.2}$$

Shock waves and the entropy condition. Let (t, x, y) be a point of jump of a unique piecewise-smooth solution to a two-dimensional Riemann problem. Condition (1.2) thus states that the vector

$$[1, S_f(u^+, u^-), S_g(u^-, u^-)],$$

where

$$S_f(u^+, u^-) \equiv \frac{f(u^+) - f(u^-)}{u^+ - u^-}, \qquad S_g(u^-, u^-) \equiv \frac{g(u^+) - g(u^-)}{u^+ - u^-}$$

is a tangent vector at the point of jump t, x, y. However, by the self-similarity of the Riemann solution, the vector

$$[1, x/t, y/t]$$

must also be tangent to the point of discontinuity. Consequently, the vector

$$[0, x/t - S_f(u^+, u^-), y/t - S_g(u^-, u^-)] \tag{2.3a}$$

is tangent to the point of discontinuity. In the plane $t = 1$, (2.3a) has the simple form

$$[0, x - S_f(u^+, u^-), y - S_g(u^-, u^-)]. \tag{2.3b}$$

Let $a > b$. Consider all planes passing through the points $(1, S_f(a, b), S_g(a, b))$ and the origin (0, 0, 0). Parametrize the straight lines thus defined in the $t = 1$ plane [these are just the lines along the vectors (2.3b)] by the angle θ, measured positive in the counterclockwise sense from the x axis, with the straight line oriented as shown in Fig. 1. The normal (pointing from the b side to the a side) to each plane is given by

$$n \equiv (n_t, n_x, n_y) = \pm\ (S_f(a, b) \tan\theta - S_g(a, b), -\tan\theta, 1), \tag{2.4}$$

where the plus sign is used for $-\pi/2 \leq \theta \leq \pi/2$ and the minus sign for $\pi/2 \leq \theta \leq 3\pi/2$.

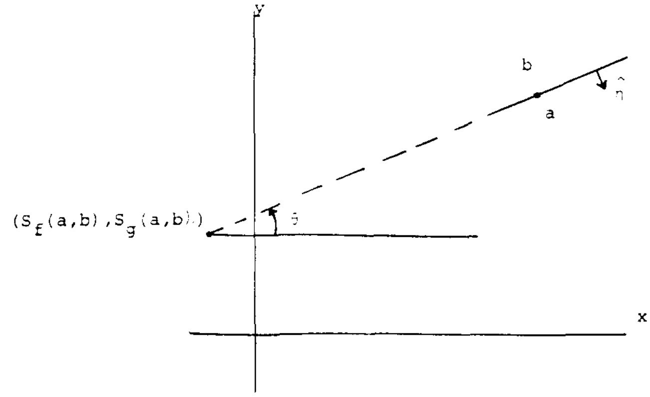

Fig. 1. The tangent line to a point of jump on a shock curve in the $t = 1$ plane. At the point the shock separates the two states a and b, $b < a$. The tangent passes through the point $(S_f(a, b), S_g(a, b))$ at angle θ with respect to the positive x axis with the orientation that the larger state a is always to the left of the shock line when viewed from the above point towards the origin.

These planes will obey the entropy condition (1.3) for the jump $a \to b$ provided

$$E(k) \equiv \pm (k - b)\{[S_g(a, b) - S_g(k, b)] - \tan\theta[S_f(a, b) - S_f(k, b)]\} \geq 0, \quad (2.5)$$

for every k such that $b \leq k \leq a$. The plus and minus signs are correlated with the angle θ as above. The following theorem summarizes this condition.

THEOREM 2.4

A point of jump $(t = 1, x, y)$, in whose neighbourhood the solution is piecewise smooth, with the one-sided limit values a and b will obey the entropy condition (1.3) if and only if its tangent vector (2.3b) is oriented so as to obey the conditions of (2.5). □

For the case $f \equiv g$ (2.5) takes on a simpler form,

$$E(k) = \pm (k - b)[S_f(a, b) - S_f(k, b)](1 - \tan\theta) \geq 0. \quad (2.6)$$

The next theorem follows directly from (2.6).

THEOREM 2.5

Let $(t = 1, x, y)$ be a point of jump of a piecewise-smooth solution to the Riemann problem for the case $f \equiv g$. Then, we have the following.

(a) If $S_f(a, b) \geq S_f(k, b)$ for every k in $[b, a]$ the point of jump will obey the entropy condition (1.3) if and only if its tangent vector (2.3b) (oriented as in Fig. 1) lies in the angular range $-3\pi/4 \leq \theta \leq \pi/4$. [Note the correlation for choice of plus and minus sign with angle as given by (2.4), which must be taken into account to obtain this result.]

(b) If $S_f(a, b) \leq S_f(k, b)$ for every k in $[b, a]$ the point of jump will obey the entropy condition (1.3) if and only if its tangent vector (2.3b) (oriented as in Fig. 1) lies in the angular range $\pi/4 \leq \theta \leq 5\pi/4$. [Note the correlation for choice of plus and minus sign with angle as given by (2.4) which must be taken into account to obtain this result.]

(c) If the states a and b are such that the quantity $S_f(a, b) - S_f(k, b)$ changes sign for some k_0 where $b < k_0 < a$, the point of jump cannot obey the entropy condition. □

Thus for $f \equiv g$, the states a and b can occur as the left and right limiting values of u at a point of jump that obeys the entropy condition if and only if either $S_f(a, b) \geq S_f(k, b)$ or $S_f(a, b) \leq S_f(k, b)$ for every k in $[a, b]$.

In two spatial dimensions, shock waves are then smooth surfaces of discontinuity, each point of which is a point of jump and satisfies the entropy condition (1.3).

The rarefaction waves. Let (t, x, y) be a regular point on a smooth level surface of a piecewise solution u to a two-dimensional Riemann problem. From (1.1) we conclude that

$$[1, f'(u), g'(u)]$$

is tangent to the surface at t, x, y. However, by the self-similarity of the Riemann solution, the vector

$$[1, x/t, y/t]$$

must also be tangent to the point. Consequently, the vector

$$[0, x/t - f'(u), y/t - g'(u)] \tag{2.7}$$

is tangent to the point on the level surface.

PROPOSITION 2.6

(a) The intersection of a smooth level surface with the plane $t = 1$ is a straight-line segment defined by the vector (2.7).

Let $u = a$ and $u = b$ denote two such level surfaces. Then

(b) if $a \neq b$, the two level surfaces cannot cross;

(c) if $a = b$, the only point that can be common to the two level surfaces and the $t = 1$ plane is $x = f'(a)$, $y = g'(a)$.

Proof. (a) This follows immediately from (2.7). The level surface is therefore a segment of a plane in t, x, y space. (b) If $a \neq b$, then the value of u on the curve along which the two level surfaces cross is not unique, contradicting the assumption that the point belongs to a level surface. (c) Follows as a consequence of the proof in part (b) and (2.7). □

A rarefaction wave is now definable in a manner analogous to the one-spatial-dimension case (see [1]). Let n be the unit normal to a smooth level surface labelled u_1 at the point $p = (t_p, x_p, y_p)$. By the smoothness of u, there exists a $p \pm \delta_p\, n$ neighbourhood of level surfaces for sufficiently small $\delta_p > 0$. For some $-\delta_p < \epsilon < \delta_p$, let u_ϵ label one such level surface. In the plane defined by the vectors n and the time axis $\hat{t}$ these level surfaces appear as straight lines segments. If the lines defined by these segments intersect at some point $(\bar{t}_\epsilon, \bar{x}_\epsilon, \bar{y}_\epsilon)$ for $\bar{t}_\epsilon < t_p$ for every $0 < |\epsilon| \leq \delta_p$ then the level surface is a *rarefaction wave*. We note that a rarefaction wave is the union of characteristic lines.

Proposition 2.6 states that two rarefaction waves (of the same value $u = r$) can intersect in the $t = 1$ plane at the point $x = f'(r)$, $y = g'(r)$. This is a reflection of Lemma 2.2, which characterizes the continuity (but possible lack of differentiability) of the solution across the plane $x = y$. This situation is pictured in Fig. 2(b) for the waves of two rarefaction fans along the plane $x = y$.

As in the one-dimensional case, a *rarefaction fan* is defined as an open set, all points of which are in rarefaction waves. A *constant state* is a domain (connected open set) in space–time t, x, y over which the solution is constant.

Explicit form of the nonlinear waves. Based on the results of Sec. 3 of [1] the forms of the nonlinear waves for the case $f \equiv g$ can be explicitly characterized. The form of a typical rarefaction fan is illustrated in Fig. 2(a) as it appears in the plane $t = 1$.

Figure 2(c) displays the three general forms of shock waves. The rarefaction fans that are used to distinguish the three forms are drawn in as dotted lines. The shock form labelled Σ_{cc} separates two regions of constant u values, denoted $u = a$, and $u = b$. The form labelled Σ_{cr} separates a region of constant u value from a fan of rarefaction lines. The Γ shock line discussed by Wagner[3] corresponds to a special case of this type of shock form. The shock labelled Σ_{rr} separates two fans of rarefaction lines.

The shock form Σ_{cc} will be a straight-line segment. The line defined by this segment will pass through the point $(S_f(a, b), S_g(a, b))$ in the x, y plane.

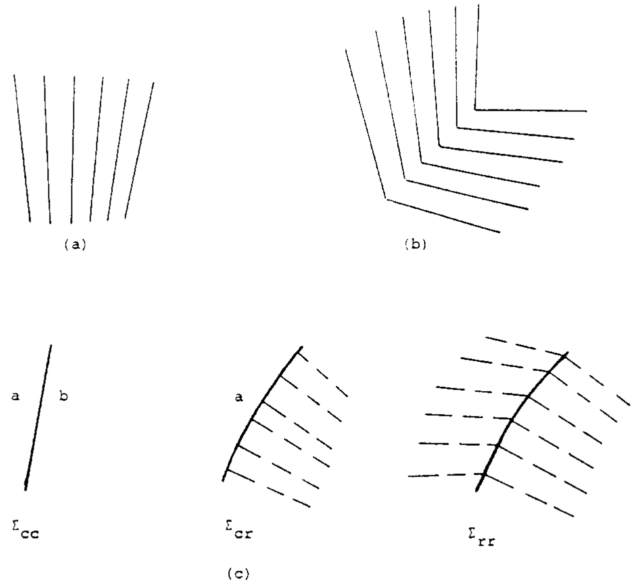

Fig. 2. (a) A typical rarefaction fan as it appears in the $t = 1$ plane. (b) Two rarefaction fans meeting continuously but not smoothly on the $x = y$ line in the $t = 1$ plane. (c) The three general forms of shock waves for the two-dimensional Riemann problem under assumptions discussed in the text. Dotted rarefaction waves are drawn to clarify the picture.

Figure 3 shows a Σ_{cr} shock separating a region of constant $u = a$ from a rarefaction fan terminating at its upper end with the value $u = b$. For clarity in the figure, we represent the rarefaction fan only by its boundary wave, $u = b$ and one interior wave. We first obtain the equation of the shock for the case $0 < \beta(r) < \pi/2$, for every r in the rarefaction fan, and for $\pi/2 < \alpha < \pi$. We parameterize the shock by the u values of the rarefaction fan. Thus the shock plane in t, x, y space has the parametric form

$$\Gamma(a, b) \equiv (t, x(r)t, y(r)t). \tag{2.8}$$

A normal to the shock surface is

$$n = (x'(r)y(r) - x(r)y'(r), y'(r), -x'(r)). \tag{2.9}$$

Letting $y(r) \equiv \gamma(r)$ be the unknown function to be solved for, we have from Fig. 3

$$\begin{aligned} x(r) &= f'(r) + (g'(r) - \gamma(r)) \tan \beta(r), \\ y(r) &= \gamma(r). \end{aligned} \tag{2.10}$$

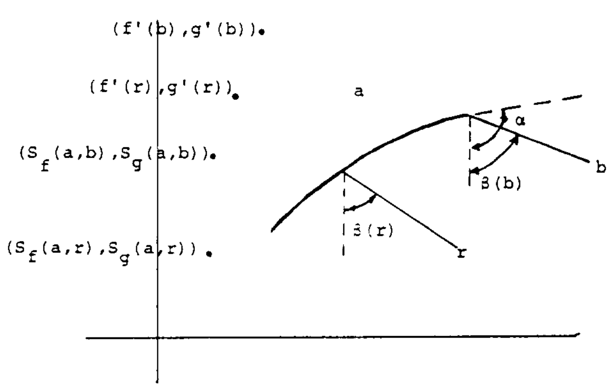

Fig. 3. The most general example of the shock type Σ_{cr}. For purposes of clarity, only two lines of the rarefaction fan that the shock separates from the constant region $u = a$ are drawn in.

The jump condition (1.2) applied to this shock gives the ordinary differential equation (for $r \neq a$) for $\gamma(r)$

$$\gamma'(r) = \frac{(\gamma(r) - S_g(r, a))[f''(r) + g''(r) \tan \beta(r) + (g'(r) - \gamma(r)) \tan' \beta(r)]}{f'(r) - S_f(r, a) + \tan \beta(r)[g'(r) - S_g(r, a)]}, \tag{2.11}$$

having boundary condition

$$\gamma(b) = \frac{S_f(a, b) + S_g(a, b) \tan \alpha - f'(b) - g'(b) \tan \beta(r)}{\tan \alpha - \tan \beta(r)}. \tag{2.12}$$

The presence of the nonlinear term due to the dependence of β on r precludes further analysis without information concerning the form of $\beta(r)$. However, if we assume all lines in the rarefaction are parallel, which in practical constructions is often the case, β becomes a constant and (2.11) reduces to

$$\gamma'(r) = \frac{(\gamma(r) - S_g(r, a))(f''(r) + g''(r) \tan \beta)}{f'(r) - S_f(r, a) + (g'(r) - S_g(r, a)) \tan \beta}. \tag{2.13}$$

Equation (2.13) has the form

$$\gamma'(r) = l(r)[\gamma(r) - S_g(r, a)],$$

thus defining $l(r)$. The solution to the system (2.12), (2.13) can be written

$$\gamma'(r) = S_g(r, a) + (\gamma(b) - S_g(b, a)) \exp\left(\int_b^r l(t)\, \mathrm{d}t\right) - \int_b^r \left(\frac{g'(z) - S_g(z, a)}{z - a}\right) \exp\left(-\int_r^z l(t)\, \mathrm{d}t\right) \mathrm{d}z. \tag{2.14}$$

We point out that our stated restriction on the angles α and β constitutes no real restriction. For any combination of angle α and β the derivation of (2.11) and solution of (2.12), (2.13) proceeds in an analogous manner. For some angles it is more advantageous to let $x(r)$ rather than $y(r)$ be the unknown function to be solved for. The cases α, $\beta = n\pi/2$ represent special cases in which the equations simplify.

The derivation of the general differential equation for shocks of type Σ_{rr} proceeds analogously to that for type Σ_{cr} but leads to a much more unmanageable equation. We therefore derive the equation for this shock type for the case in which the lines in each rarefaction fan are parallel to one another (see Fig. 4).

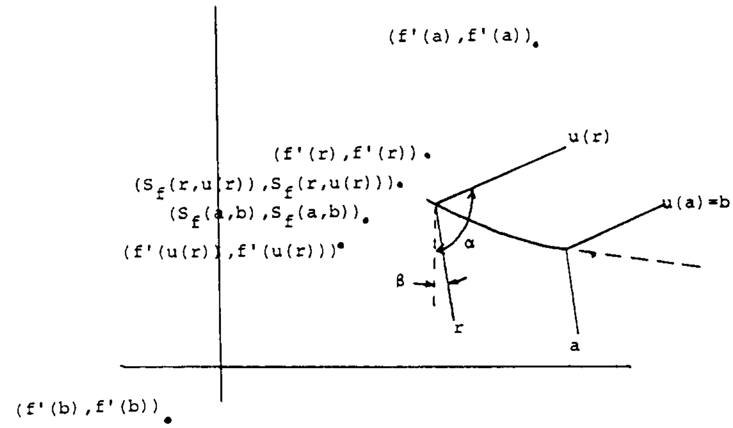

Fig. 4. A particular example of shock type Σ_{rr} where the two rarefaction fans on either side are composed of separately parallel lines. For clarity, only two of the lines in each rarefaction have been drawn in.

We again parametrize the shock plane by the u values of one of the two rarefaction fans (say for the lower fan pictured in Fig. 4). Equations (2.8) and (2.9) still hold, but now

$$x(r) = \frac{g'(u(r)) - g'(r) + f'(u(r)) \cot \alpha - f'(r) \cot \beta}{\cot \alpha - \cot \beta}, \qquad (2.15)$$
$$y(r) = \frac{f'(u(r)) - f'(r) + g'(u(r)) \tan \alpha - g'(r) \tan \beta}{\tan \alpha - \tan \beta}.$$

Substitution of (2.15) in (2.9) and application of the jump condition (1.2) gives an equation for $u(r)$:

$$u'(r)[g''(u(r)) \tan \alpha + f''(u(r))][f'(r) - S_f(u(r), r) + (g'(r) - S_g(u(r), r)) \tan \beta]$$
$$= [g''(r) \tan \beta + f''(r)][f'(u(r)) - S_f(u(r), r) + (g'(u(r)) - S_g(u(r), r)) \tan \alpha], \quad (2.16)$$

having the boundary condition $u(a) = b$ (see Fig. 4) which will be a known value.

For the case $f = g$, the angles drop out of (2.16), giving

$$u'(r) = \frac{f''(r)}{f''(u(r))} \frac{f'(u(r)) - S_f(u(r), r)}{f'(r) - S_f(u(r), r)}. \qquad (2.17)$$

No solution is possible to (2.16) or (2.17) without knowing the form of f. If we assume $f(r) = lr^2$, the solution to (2.17) is $u(r) = -r + a + b$ where a and b are as defined in Fig. 4. Substitution of the solution to (2.16) or (2.17) in (2.15) gives the parametric equation for the shock curve.

As a final comment we note that a rarefaction line $u = a$ will terminate tangentially on a shock (Σ_{cr} or Σ_{rr}) in the plane $t = 1$ at a point p where the limiting u values on each side of the shock wave are $u = a$ and $u = b$ only if the point $(f'(a), g'(a))$ coincides with the point $(S_f(a, b), S_g(a, b))$.

In the half-spaces $x < y$, $x > y$, irregular points in the solution occur as intersection curves of shock surfaces, and at the discontinuities in the initial data in the plane $t = 0$. In the $t = 1$ plane these irregular points are therefore points at which N (≥ 3) shocks meet. The line $x = y$ may also contain irregular points, each corresponding to a rarefaction wave $u = a$ in the $x > y$ half-plane meeting a rarefaction wave $u = a$ in the $x < y$ half-plane continuously but not smoothly. Irregular points corresponding to the meeting of two shock waves (one from each half-plane) can also appear on this line. A third possibility is the termination of a shock wave on the line $x = y$. For this to occur the shock wave must terminate with zero strength.

3. TWO-DIMENSION METHOD OF SOLUTION CONSTRUCTION—EXAMPLES

Construction of solutions to two-dimensional Riemann problems was first approached by Guckenheimer[2], who gave the solutions to two example problems. Additionally, he listed several of the principles concerning the restriction of the solution in the plane $t = 1$. Wagner[3] has constructed solutions to (1.1) for the case that the functions f and g are both convex (analogously both concave) and that the initial data is constant in the four quadrants of the (x, y) plane. For the case $f \equiv g$, his solutions obey the entropy condition (1.3). He demonstrates that if f and g are "sufficiently close," additional restrictions on the derivatives of the functions f and g are required for the $f \equiv g$ solutions to hold for the $f \neq g$ case.

Having classified the most general nonlinear waves that can appear, the construction method these two papers have introduced can be used to construct the entropy-obeying solution to the general 2-D Riemann problem.

The method of solution construction is the following. The construction is done in the $t = 1$ plane. The curve $(f'(u), g'(u))$ and the set of points $((S_f(v, w), S_g(v, w))$, which dictate the orientation of rarefaction lines and shock curves, are identified. The nonlinear waves and their analytic forms, and the irregular points from which the solutions will be composed, are identified.

The angle restrictions which the entropy condition imposes on the tangent vectors to the nonlinear waves are obtained. In practice this imposes limits on the extent to which a particular shock-wave form may appear in the plane in a particular region and determines the "fitting together" of the various nonlinear elements. The solution in the region $x^2 + y^2 \geq a^2$ for some a is obtained as the solution of noninteracting one-dimensional Riemann problems. The solution is then extrapolated into the region $x^2 + y^2 < a^2$ by fitting together nonlinear waves. In the following subsections we display the entropy-obeying solutions for a few classes of Riemann problems for the cases $f \equiv g$.

For the case $f \equiv g$ the nonlinear wave forms and the irregular points of the solution follow from Sec. 2 and [1]. This analysis also holds for functions f and g which differ by a multiplicative constant, since this can be rotated into a problem $\bar{f} \equiv \bar{g}$. We note that Example 1 of [2] falls into this category.

3.1 *A convex f example*

We illustrate the case for f having no inflection points with initial data consisting of three wedges of constant states a, b and c ($a < b < c$) centered on the origin in the $x \geq y$ half of the plane. Figure 5 shows the six unique (i.e. 'topologically different') entropy-obeying solutions.

By Theorem 2.5(a) it follows that the solutions satisfy the entropy condition. By this theorem the three wedges can be changed in size relative to one another anywhere in the range $-3\pi/4 \leq \theta \leq \pi/4$ and the solutions will not change their topological form.

Completely analogous solutions and conclusions follow from Theorem 2.5(b) for the concave case of $f \equiv g$.

Shocks of type Σ_{rr} do not appear in the solutions in Fig. 5. For them to appear in the case of convex f it is necessary to go to initial data consisting of four wedges in the half-plane $x \geq y$. Twenty-four "topologically distinct" solutions exist, one of which, showing a shock of type Σ_{rr}, we sketch in Fig. 6.

3.2 *A single-inflection-point example*

We now consider the solution for $f \equiv g$ obeying the properties

(1) f: $[a, b] \to [c, d]$, $a < b$, $c < d$, $a, b, c, d \in \mathbf{R}$;
(2) f is strictly monotonic on $[a, b]$;
(3) f has a single inflection point at u_i in (a, b);
(4) $f''(u) > 0$ on $[a, u_i)$, $f''(u) < 0$ on $(u_i, b]$.

Without loss of generality we shall take $a = 0$, $b = 1$, $c = 0$, $d = 1$. f has the property that for every $u \in [0, 1]$, there exists a unique $u^* \in [0, 1]$ such that $f'(u^*) = S_f(u, u^*)$. The meaning of the iterated notation u^{**} is then to be understood as $(u^*)^*$.

Shocks of type Σ_{rr} separating a fan of rarefaction lines of values $u > u_i$ from a fan of values $u < u_i$ appear in the solution to the two-dimensional Riemann problem for this f function. If a rarefaction wave has value $u = r$ on one side of the shock, the rarefaction wave leaving on the other side will have the value $u = r^*$ (defined above) and will leave the shock line tangentially. The form derived for shocks of type Σ_{rr} in Sec. 2 does not apply since there it was assumed that the angles α and β of the two rarefaction fans were known; consequently the shape of the shock in that analysis was found by determining the state values of the rarefaction lines on one side given the state values on the other. Here knowledge of the states on either side of the shock, and the implication that one set of rarefactions leave tangentially, determines the shape of the shock curve. Therefore we determine parametrically the shape of this shock type and show that the shock obtained obeys the entropy condition (1.3) for the case $f \equiv g$.

Consider the Riemann problem shown in Fig. 7(a) with w and v as in Fig. 7(b). The solution is shown in Fig. 7(c) where the shock of type Σ_{rr} is labelled $\Psi(r, r^*)$. The analysis for the form of Ψ proceeds as in Sec. 2. Parametrize Ψ as in (2.8) with

$$x(r) = \psi(r), \qquad (3.1)$$
$$y(r) = f'(r) + (f'(r) - \psi(r)) \cot \alpha(r).$$

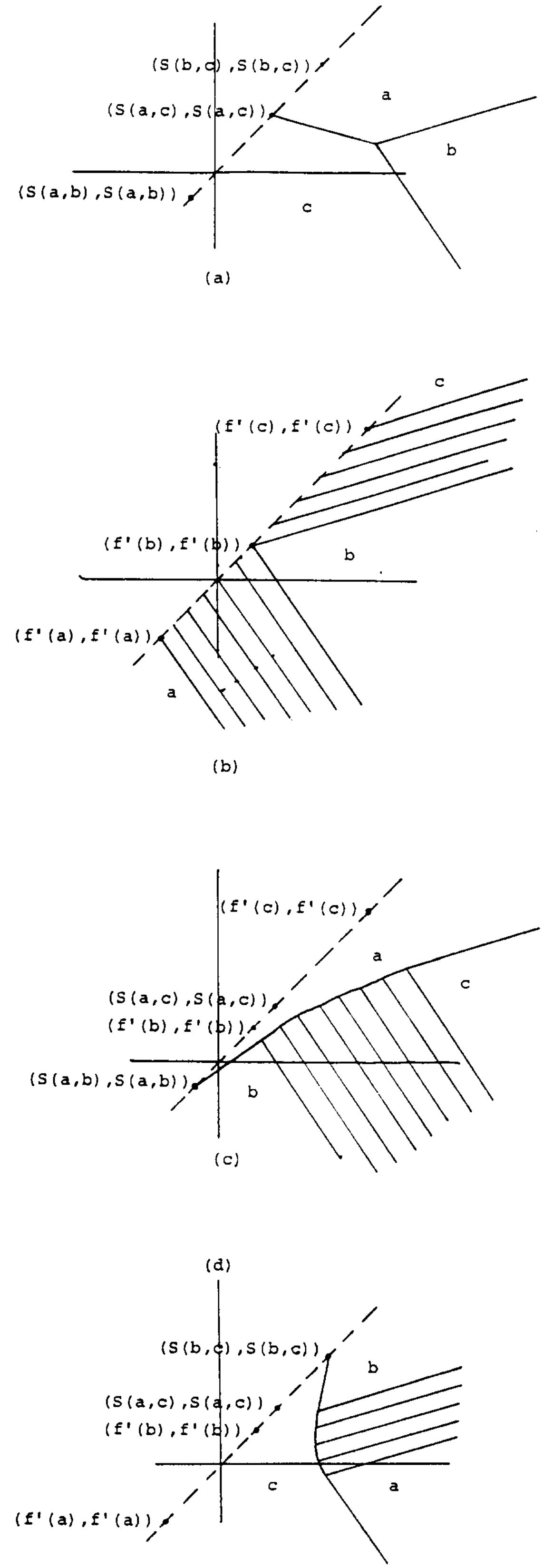

Fig. 5. The six qualitatively unique entropy-obeying solutions in the region $x \geq y$ to (1.1) for $f_1 \equiv f_1 \equiv f$, f having no inflection points, with initial data composed of three wedges of constant u value ($a < b < c$) centered on (0, 0).

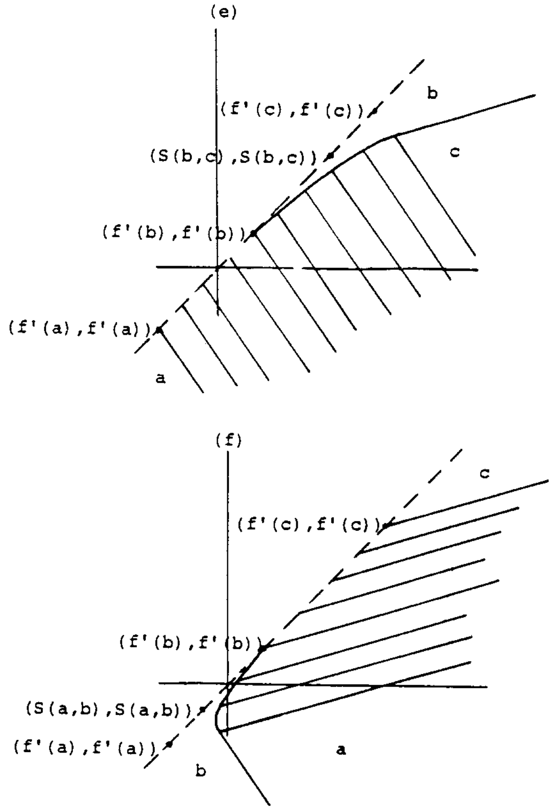

Fig. 5. *(Continued)*.

Application of the jump condition (1.2) gives the equation for ψ:

$$\psi'(r) = \frac{[f''(r)(1 + \cot\alpha(r)) + (\psi(r) - f'(r))\cot'(\alpha(r))]}{(f'(r) - S_f(r, r^*))(1 + \cot\alpha(r))}, \tag{3.2}$$

with boundary condition

$$\psi(v) = \frac{f'(v)(1 + \cot\alpha(v)) - f'(v^*)(1 + \cot\beta(v^*))}{\cot\alpha(v) - \cot\beta(v^*)}. \tag{3.3}$$

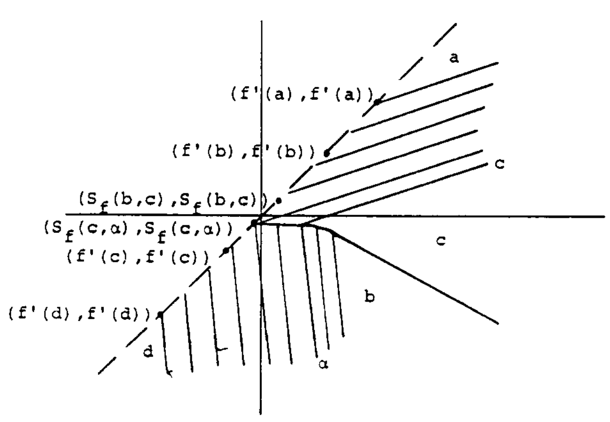

Fig. 6. A sketch of a four wedge initial data solution to (1.1) showing the appearance of a Σ_{rr} shock. The Σ_{rr} shock has been drawn as a straight line which would be the case for $f(u) = u^2/2$. Note that the Σ_{rr} shock segment joins smoothly to the Σ_{cr} segment with the same slope. The value α can be determined from the intersection of the Σ_{cr} shock and the rarefaction line $u = c$. In this example the states are ordered $d < c < b < a$.

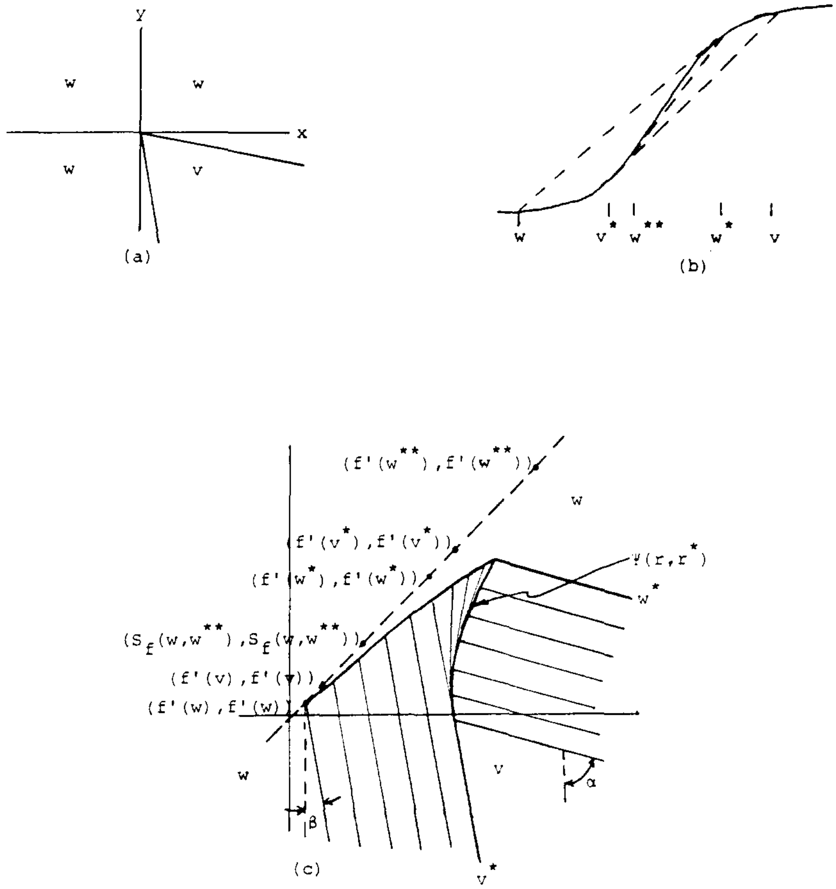

Fig. 7. A two-dimensional Reimann problem for the function f described in Sec. 3.2. (a) The initial data. (b) The function f defining various states found in the solution. (c) The solution.

As in (2.11), variation of α along the rarefaction introduces a nonlinear term into (3.2). If we assume all lines in this rarefaction are parallel [as indeed have been drawn in Fig. 7(c)], which will be the case in the solutions we display in this paper, (3.2) simplifies greatly to

$$\psi'(r) = f''(r)\,\frac{(\psi(r) - S_f(r, r^*))}{(f'(r) - S_f(r, r^*))}, \tag{3.4}$$

having the solution

$$\psi(r) = S_f(r, r^*) + [\psi(v) - S_f(v, v^*)] \exp \int_v^r l(t)\,\mathrm{d}t - \int_v^r \left(\exp \int_z^r l(t)\,\mathrm{d}t\right) \frac{\partial S_f(z, z^*(z))}{\partial z}\,\mathrm{d}z, \tag{3.5}$$

where

$$l(t) = \frac{f''(t)}{f'(t) - S_f(t, t^*)}. \tag{3.6}$$

Note that given r in $[w^*, v]$, the form of the function f guarantees the existence of r^* and for every k such that $r^* \le k \le r$ we have $S_f(r, r^*) \le S_f(k, r^*)$. From the construction, it is clear that the above shock obeys the conditions of Theorem 2.5(b) and hence obeys the entropy condition.

We return briefly to Fig. 7(c) to discuss the other curved shock, which is composed of two shocks of type Σ_{cr}, one separating the fan of parallel lines $u = w \to u = v^*$ from the constant region $u = w$, the other separating a fan of nonparallel lines $u = v^* \to u = w^{**}$ from the constant region $u = w$. The two shocks have the same tangent line where they meet. The construction and Theorem 2.5(a) guarantee that this composite shock obeys the entropy condition.

In Fig. 8 we present the entropy obeying solutions to (1.1) for $f \equiv g$ with f as described above for initial data of three wedges of constant states a, b and c ($a < b < c$) centered on the origin. We consider only the two general cases: either $a < u_i < b < c$ or $a < b < u_i < c$, otherwise the problem reduces to the purely convex or concave case discussed in Sec. 3.1. By the qualitative symmetry of the function f about its inflection point, there are only six

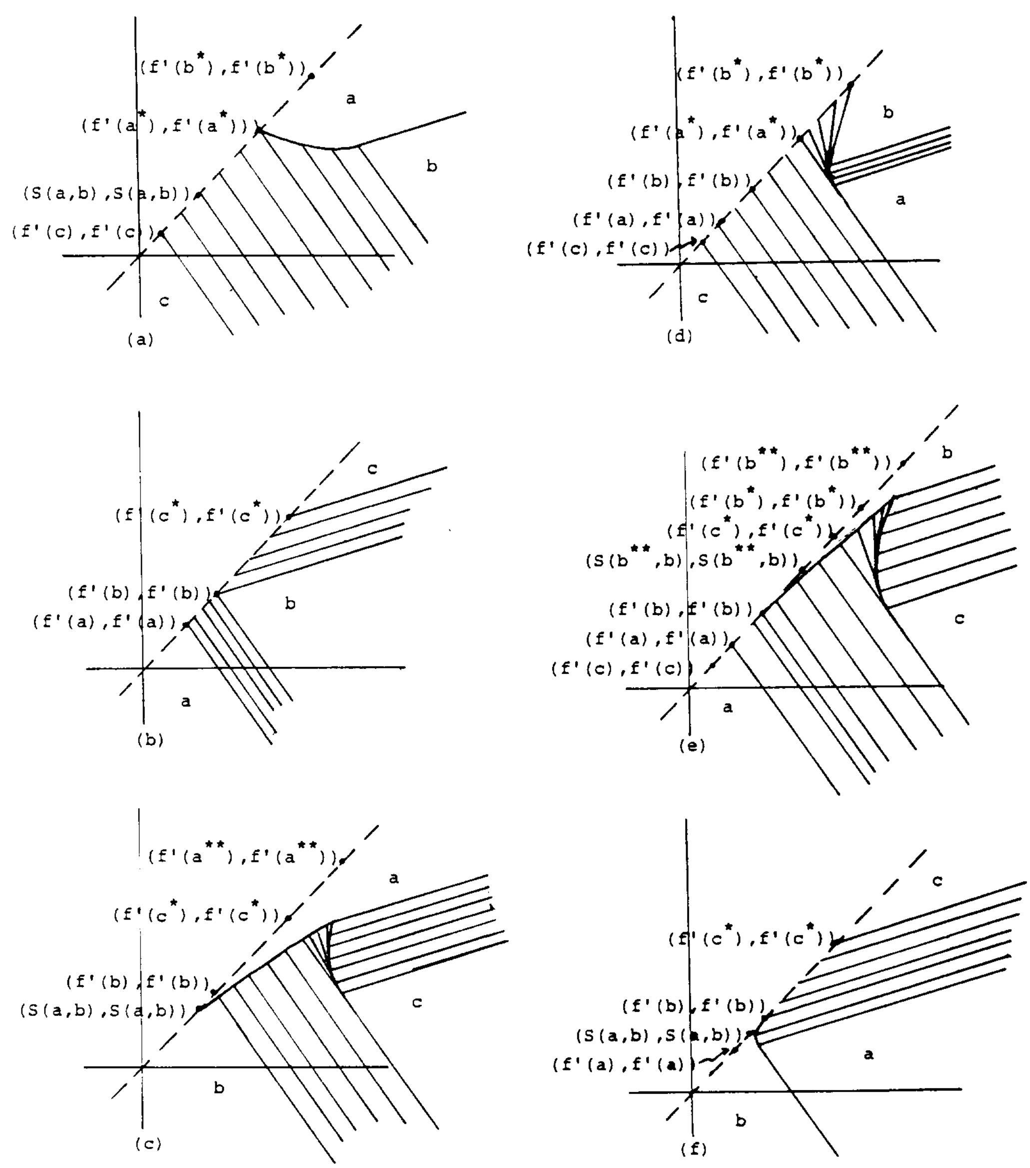

Fig. 8. The six qualitatively distinct solutions in the $x \geq y$ half-plane for the three-wedge Riemann problem discussed in Sec. 3.2. Note that one of the three initial data states a, b and c is separated from the other two by u_i, the inflection point of f.

qualitatively distinct solutions. As previously mentioned, the solutions obtained in the $x \geq y$ half of the plane can be obtained independently of the solutions in the $x \leq y$ half.

4. THE TWO-DIMENSIONAL RIEMANN PROBLEM FOR TWO-PHASE FLOW IN POROUS MEDIA

For incompressible two-phase flow in a porous medium, with capillarity effects and gravity ignored,† the system of equations to be solved in a source-free region is

$$\frac{\partial s}{\partial t} + \mathbf{v}(s, P) \cdot \nabla f(s) = 0, \tag{4.1a}$$

$$\nabla \cdot \mathbf{v}(s, P) = 0, \tag{4.1b}$$

where s is the volume fraction (saturation) of one of the two phases (the other fraction being $1 - s$) and p is the pressure field in the porous medium. The field $\mathbf{v}$ is the total fluid (consisting of both phases) velocity. The velocity $\mathbf{v}$ is generally assumed to be proportional to ∇P (Darcy's Law) and when the form of this relation is specified the system (4.1) can be solved. A form commonly used for $f(s)$ is (immiscible flow[6])

$$f(s) = \frac{s^2}{s^2 + r(1 - s)^2}, \tag{4.2}$$

where r is the ratio of the two phase viscosities (we assume constant viscosities). The fractional flow curve given above then has all the properties of the f function described in Sec. 3.2.

One method of solving (4.1) numerically consists of solving (4.1a) and (4.1b) sequentially. Thus for purposes of solving the hyperbolic equation (4.1a) the velocity field can be taken as some given vector field $\mathbf{v}(\mathbf{x})$. For two-dimensional problems, the hyperbolic problem is therefore

$$\frac{\partial s}{\partial t} + v_x(x, y) \frac{\partial f(s)}{\partial x} + v_y(x, y) \frac{\partial f(s)}{\partial y} = 0. \tag{4.3}$$

Shocks in two-phase systems are very rapid (in this model, discontinuous) changes in the phase, from a region that has largely phase 1 in its pore spaces to that having largely phase 2. To be specific we will call the phases oil and water, with $s = 0$ corresponding to pure oil and $s = 1$ corresponding to pure water. The system (4.1) describes a secondary oil-recovery procedure where injected water is used to force reservoir oil towards production wells. The oil–water shock surface is typically called a "bank" (water bank if the water is displacing the oil, oil bank if the oil is displacing water). We can therefore study the dynamics involved in the interaction of two such banks, for example when the water banks from two separate injection wells meet, by approximating the problem as a Riemann problem.

Figure 9(a) depicts the problem to be solved and Fig. 9(b) its approximation as a Riemann problem. Let v denote a value of v_x averaged over the interaction area; similarly let w be an averaged value of v_y. Equation (4.3) becomes

$$\frac{\partial s}{\partial t} + v \frac{\partial f(s)}{\partial x} + w \frac{\partial f(s)}{\partial y} = 0. \tag{4.4}$$

This is now of the form (1.1) with f and g differing by a multiplicative constant. We can therefore solve (4.4) in rotated coordinates $\bar{x}$, $\bar{y}$ where $\bar{f} = \bar{g}$.

Figure 9(c) displays the solution to the Riemann problem of Fig. 9(b) in some rotated ξ, η system (the rotation of course depends on the relative strengths of v and w; we have assumed $v > 0$, $w > 0$). Figure 9(d) displays the solution when a and c are interchanged.

†We have also set the rock porosity function $\phi(\mathbf{x}) = 1$.

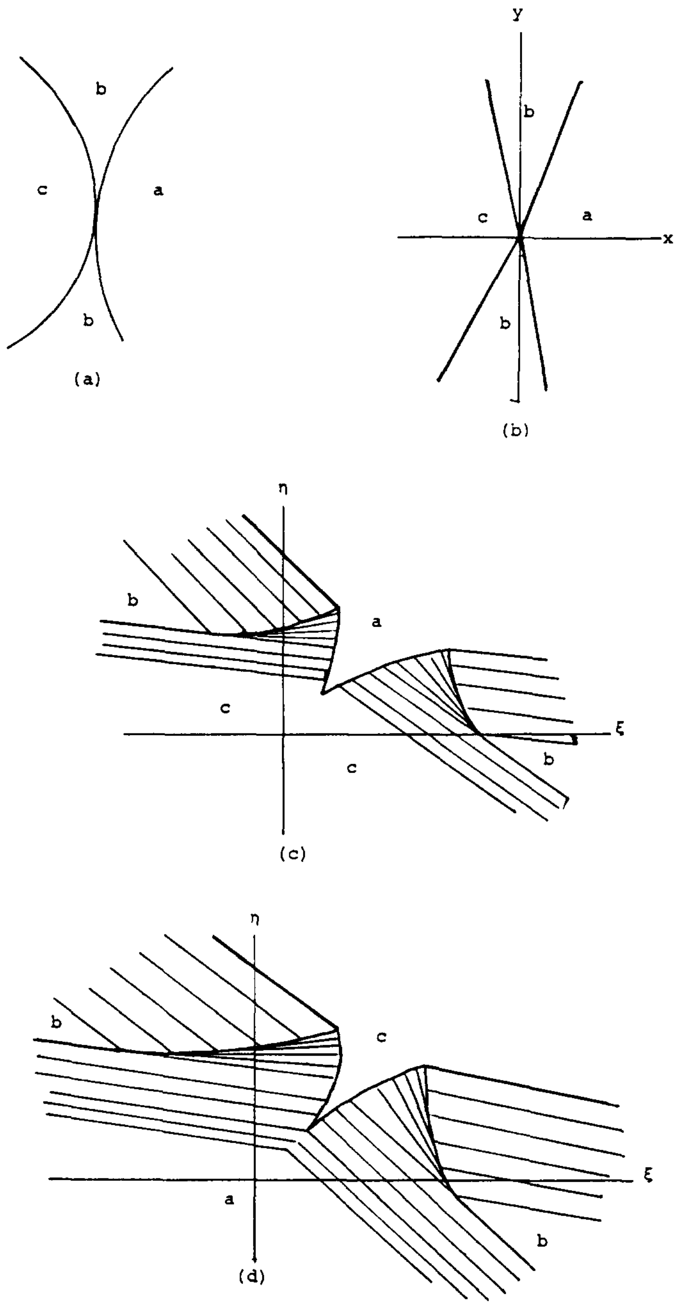

Fig. 9. (a) The interaction of two water banks in a two-dimensional oil reservoir. (b) The Riemann problem approximation to the interaction. (c,d) The solutions to the Riemann problem approximation for two example situations.

We note that for flow with gravity, the flux function associated with two-phase flow has two inflection points, as opposed to a single inflection point for the gravity-free-flow case.

5. CONCLUDING REMARKS

The general solution to the two-dimensional Riemann problem for the case $f_1 \equiv f_2 \equiv f$, $f \in C^2 : R \rightarrow R$, f having at most one inflection point, has been shown to be characterizable in terms of nonlinear (rarefaction and shock) waves in a manner analogous to the one-dimensional Riemann problem. We believe this characterization in terms of nonlinear waves is the correct formulation in which to approach the construction of solutions to two-dimensional Riemann problems for the general case $f_1 \neq f_2$. We conjecture that the set of rarefaction and shock waves described in this paper will provide the complete solutions for many cases $f_1 \neq f_2$. A regularity theorem giving conditions under which all two-dimensional Riemann problem solutions are piecewise smooth would immediately give greater generality of application to the nonlinear waveforms given here.

These solutions to two-dimensional Riemann problems also supply a set of problems for the testing of finite-difference schemes. The richness of structure of these solutions lends itself to this purpose.

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